Alan Turing's wartime note to Ralph Noskwith

Item AMT-D-19 in the archive at King's College, Cambridge, is a manuscript note by Alan Turing on a problem in n-dimensional geometry. It was given to his wartime colleague Ralph Noskwith. The note was acquired by the Archive in 2023 and can be seen on-line in the Turing Digital Archive.

The Archive describes it as a 'quick-and-dirty explanation' of the problem. But it is not quick!

Turing considers a unit equilateral n-simplex with vertices given by n-vectors

$$l_0 = 0, l_1, \dots l_n \, .$$

Thus $l_i^2 = 1, (l_i - l_j)^2 = 1$ for $i \neq j$, and so $l_i . l_j = \frac{1}{2}$, where the usual Euclidean metric and dot product is used. Turing gives explicit coordinates for the l_i but these are not needed.

Turing then considers the lattice of points $\sum_1^n m_i l_i$, where the m_i are integers. The problem is: how many of them are at unit distance from the origin?

Clearly $\pm l_i$ and $l_i - l_j$ for $i \neq j$, are such points, and these $n(n+1)$ points are in fact the only such. Simply note that by using the inner product values stated above,

$$(\sum_i m_i l_i)^2 = \sum_i m_i^2 + \sum_{i<j} m_i m_j = \frac{1}{2}(\sum_i m_i)^2 + \frac{1}{2}(\sum_i m_i^2)$$

But this expression can only equal 1 in the cases already identified. It does not need an induction proof such as Turing gives.

Andrew Hodges, May 2023.